Existence of Triangular Lie Bialgebra Structures II

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Dedicated to the memory of my father

Abstract

We characterize finite-dimensional Lie algebras over an arbitrary field of characteristic zero which admit a non-trivial (quasi-) triangular Lie bialgebra structure.

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§1. Introduction

In general, a complete classification of all triangular Lie bialgebra structures is very difficult. Nevertheless, Belavin and Drinfel'd succeeded in [2] to obtain such a classification for *every* finite-dimensional *simple* Lie algebra over the complex numbers. The aim of this paper is much more modest in asking when there exist *non-trivial* (quasi-) triangular Lie bialgebra structures. Michaelis showed in [11] that the existence of a two-dimensional non-abelian subalgebra implies the existence of a non-trivial triangular Lie bialgebra structure over any ground field of arbitrary charactristic. In [6] we used a slight generalization of the main result of [11] (cf. also [2, Section 7]) in order to establish a non-trivial triangular Lie bialgebra structure on almost any finite-dimensional Lie algebra over an algebraically closed field of arbitrary characteristic. Moreover, we obtained a characterization of those finite-dimensional Lie algebras which admit non-trivial (quasi-) triangular Lie bialgebra structures. In this paper we extend the results of [6] and [7] to arbitrary ground fields of characteristic zero. A crucial result in the first part of this paper is [6, Theorem 1] which is not available for non-algebraically closed fields and has to be replaced by a more detailed analysis. As in the previous papers, it turns out that with the exception of a few cases occurring in dimension three, every finite-dimensional non-abelian Lie algebra over an arbitrary field of characteristic zero admits a *non-trivial triangular* Lie bialgebra structure. As a consequence we also obtain that every finite-dimensional non-abelian Lie algebra over an arbitrary field of characteristic zero has a *non-trivial coboundary* Lie bialgebra structure. The latter extends the main result of [4] from the real and complex numbers to arbitrary ground fields of characteristic zero.

Let us now describe the contents of the paper in more detail. In Section 2 we introduce the necessary notation and prove some preliminary results reducing the existence of non-trivial triangular Lie bialgebra structures to three-dimensional Lie algebras or in one case showing at least that the derived subalgebra is abelian of dimension at most two. The next section is entirely devoted to the

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existence of non-trivial (quasi-) triangular Lie bialgebra structures on three-dimensional *simple* Lie algebras. It is well-known that every three-dimensional simple Lie algebra is the factor algebra of the Lie algebra associated to a quaternion algebra modulo its one-dimensional center. Since the Lie bracket of these so-called quaternionic Lie algebras are explicitly given, one can compute the solutions of the classical Yang-Baxter equation with invariant symmetric part which generalizes [6, Example 1]. This enables us to prove that a three-dimensional simple Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic not two admits a non-trivial (quasi-) triangular Lie bialgebra structure if and only if it is isomorphic to the split three-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$. Moreover, we observe that the classical Yang-Baxter operator for every three-dimensional simple Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic not two is closely related to the determinant. In the last section we finally prove the characterization of those finite-dimensional Lie algebras which admit *non-trivial (quasi-) triangular* Lie bialgebra structures and show that every finite-dimensional non-abelian Lie algebra over an arbitrary field of characteristic zero admits a *non-trivial coboundary* Lie bialgebra structure.

§2. Preliminaries

Let $\mathbb{F}$ be a commutative field of arbitrary characteristic. A *Lie coalgebra* over $\mathbb{F}$ is a vector space $\mathfrak{c}$ over $\mathbb{F}$ together with a linear transformation

$$\delta : \mathfrak{c} \longrightarrow \mathfrak{c} \otimes \mathfrak{c},$$

such that

$$\mathrm{Im}(\delta) \subseteq \mathrm{Im}(\mathrm{id}_{\mathfrak{c}} - \tau), \tag{1}$$

and

$$(\mathrm{id}_{\mathfrak{c}} + \xi + \xi^2) \circ (\mathrm{id}_{\mathfrak{c}} \otimes \delta) \circ \delta = 0, \tag{2}$$

where $\mathrm{id}_{\mathfrak{c}}$ denotes the identity mapping on $\mathfrak{c}$, $\tau : \mathfrak{c} \otimes \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c}$ denotes the *switch mapping* sending $x \otimes y$ to $y \otimes x$ for every $x, y \in \mathfrak{c}$, and $\xi : \mathfrak{c} \otimes \mathfrak{c} \otimes \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c} \otimes \mathfrak{c}$ denotes the *cycle mapping* sending $x \otimes y \otimes z$ to $y \otimes z \otimes x$ for every $x, y, z \in \mathfrak{c}$. The mapping δ is called the *cobracket* of $\mathfrak{c}$, (1) is called *co-anticommutativity*, and (2) is called the *co-Jacobi identity*. Note that any cobracket on a one-dimensional Lie coalgebra is the zero mapping since $\mathrm{Im}(\mathrm{id} - \tau) = 0$. This is dual to the statement that every bracket on a one-dimensional Lie algebra is zero. For further information on Lie coalgebras we refer the reader to [10] and the references given there.

A *Lie bialgebra* over $\mathbb{F}$ is a vector space $\mathfrak{a}$ over $\mathbb{F}$ together with linear transformations $[\cdot,\cdot] : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$ and $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ such that $(\mathfrak{a}, [\cdot,\cdot])$ is a Lie algebra, $(\mathfrak{a}, \delta)$ is a Lie coalgebra, and δ is a *derivation* from the Lie algebra $\mathfrak{a}$ into the $\mathfrak{a}$-module $\mathfrak{a} \otimes \mathfrak{a}$, i.e.,

$$\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x) \qquad \forall\ x, y \in \mathfrak{a},$$

where the tensor product $\mathfrak{a} \otimes \mathfrak{a}$ is an $\mathfrak{a}$-module via the *adjoint diagonal action* defined by

$$x \cdot \left(\sum_{j=1}^{n} a_j \otimes b_j \right) := \sum_{j=1}^{n} ([x, a_j] \otimes b_j + a_j \otimes [x, b_j]) \qquad \forall\ x, a_j, b_j \in \mathfrak{a}$$

(cf. [3, Section 1.3A]). A Lie bialgebra structure $(\mathfrak{a}, \delta)$ on a Lie algebra $\mathfrak{a}$ is called *trivial* if $\delta = 0$.

A *coboundary Lie bialgebra* over $\mathbb{F}$ is a Lie bialgebra $\mathfrak{a}$ such that the cobracket δ is an *inner derivation*, i.e., there exists an element $r \in \mathfrak{a} \otimes \mathfrak{a}$ such that

$$\delta(x) = x \cdot r \qquad\qquad \forall\, x \in \mathfrak{a}$$

(cf. [3, Section 2.1A]).

In order to describe those tensors r which give rise to a coboundary Lie bialgebra structure, we will need some more notation. For $r = \sum_{j=1}^{n} r_j \otimes r_j' \in \mathfrak{a} \otimes \mathfrak{a}$ set

$$r^{12} := \sum_{j=1}^{n} r_j \otimes r_j' \otimes 1, \quad r^{13} := \sum_{j=1}^{n} r_j \otimes 1 \otimes r_j', \quad r^{23} := \sum_{j=1}^{n} 1 \otimes r_j \otimes r_j',$$

where 1 denotes the identity element of the universal enveloping algebra $U(\mathfrak{a})$ of $\mathfrak{a}$. Note that the elements r^{12}, r^{13}, r^{23} are considered as elements of the associative algebra $U(\mathfrak{a}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{a})$ via the canonical embedding $\mathfrak{a} \hookrightarrow U(\mathfrak{a})$. Therefore one can form the commutators given by

$$\begin{aligned}
[r^{12}, r^{13}] &= \sum_{i,j=1}^{n} [r_i, r_j] \otimes r_i' \otimes r_j', \\
[r^{12}, r^{23}] &= \sum_{i,j=1}^{n} r_i \otimes [r_i', r_j] \otimes r_j', \\
[r^{13}, r^{23}] &= \sum_{i,j=1}^{n} r_i \otimes r_j \otimes [r_i', r_j'].
\end{aligned}$$

Then the mapping

$$\mathrm{CYB} : \mathfrak{a} \otimes \mathfrak{a} \longrightarrow \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$$

defined via

$$r \longmapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$$

is called the *classical Yang-Baxter operator* for $\mathfrak{a}$. The equation $\mathrm{CYB}(r) = 0$ is the *classical Yang-Baxter equation* (CYBE) for $\mathfrak{a}$, and a solution of the CYBE is called a *classical r-matrix* for $\mathfrak{a}$ (cf. [3, Section 2.1B]).

Assume for the moment that the characteristic of the ground field $\mathbb{F}$ is not two. For any vector space V over $\mathbb{F}$ and every natural number n the *symmetric group* S_n of degree n acts on the n-fold tensor power $V^{\otimes n}$ of V via

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in S_n; v_1, \ldots, v_n \in V.$$

The $\mathbb{F}$-linear transformation $\mathcal{S}_n : V^{\otimes n} \to V^{\otimes n}$ defined by $t \mapsto \sum_{\sigma \in S_n} \sigma \cdot t$ is called the *symmetrization transformation*. The elements of the image $\mathrm{Im}(\mathcal{S}_n)$ of $\mathcal{S}_n$ are just the *symmetric* n-tensors, i.e., elements $t \in V^{\otimes n}$ such that $\sigma \cdot t = t$ for every $\sigma \in S_n$. Moreover, $\mathrm{Im}(\mathcal{S}_n)$ is canonically isomorphic to the n-th *symmetric power* $S^n V$ of V. The $\mathbb{F}$-linear transformation $\mathcal{A}_n : V^{\otimes n} \to V^{\otimes n}$ defined by $t \mapsto \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)(\sigma \cdot t)$ is called the *skew-symmetrization* (or *alternation*) *transformation*. The elements of the image $\mathrm{Im}(\mathcal{A}_n)$ of $\mathcal{A}_n$ are just the *skew-symmetric* n-tensors, i.e., elements $t \in V^{\otimes n}$ such that $\sigma \cdot t = \mathrm{sign}(\sigma) t$ for every $\sigma \in S_n$. Since $\mathrm{Im}(\mathcal{A}_n)$ is canonically isomorphic to the n-th *exterior power* $\Lambda^n V$ of V, in the following we will always identify skew-symmetric n-tensors with elements of $\Lambda^n V$; e.g., we write

$$v_1 \wedge v_2 = (\mathrm{id}_V - \tau)(v_1 \otimes v_2) = v_1 \otimes v_2 - v_2 \otimes v_1$$

in the case $n = 2$, where $\tau : V^{\otimes 2} \to V^{\otimes 2}$ is given by $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$, and

$$v_1 \wedge v_2 \wedge v_3 = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$$

in the case $n = 3$. As a direct consequence of our identifications, we also have that

$$V^{\otimes 2} = S^2 V \oplus \Lambda^2 V.$$

If $\mathfrak{a}$ is a Lie algebra and M is an $\mathfrak{a}$-module, then the set of $\mathfrak{a}$-*invariant elements* of M is defined by

$$M^{\mathfrak{a}} := \{m \in M \mid a \cdot m = 0 \quad \forall\, a \in \mathfrak{a}\}.$$

Let $r \in \mathfrak{a} \otimes \mathfrak{a}$ and define $\delta_r(x) := x \cdot r$ for every $x \in \mathfrak{a}$. Obviously, a coboundary Lie bialgebra structure $(\mathfrak{a}, \delta_r)$ on $\mathfrak{a}$ is *trivial* if and only if $r \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$. Moreover, Drinfel'd observed that δ_r defines a Lie bialgebra structure on $\mathfrak{a}$ if and only if $r + \tau(r) \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ and $\mathrm{CYB}(r) \in (\mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ (see [5, Section 4, p. 804] or [3, Proposition 2.1.2]). In particular, every solution r of the CYBE satisfying $r + \tau(r) \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ gives rise to a coboundary Lie bialgebra structure on $\mathfrak{a}$. Following Drinfel'd such Lie bialgebra structures are called *quasi-triangular*, and quasi-triangular Lie bialgebra structures arising from skew-symmetric classical r-matrices are called *triangular*.

In [6, Example 2] we already observed that the three-dimensional *Heisenberg algebra* does *not* admit *any* non-trivial triangular Lie bialgebra structure. Let us recall that the (non-abelian) *nilpotent* three-dimensional *Heisenberg algebra*

$$\mathfrak{h}_1(\mathbb{F}) = \mathbb{F}p \oplus \mathbb{F}q \oplus \mathbb{F}\hbar$$

is determined by the so-called *Heisenberg commutation relation*

$$[p, q] = \hbar.$$

Let us conclude this section by several preliminary results which will be used in the proof of the main theorem of this paper. The first lemma is already contained in [7] and is an immediate consequence of the arguments in the proofs of [6, Theorem 2 and Theorem 3].

Lemma 1. *Let $\mathfrak{a}$ be a finite-dimensional non-abelian Lie algebra over an arbitrary field $\mathbb{F}$ with non-zero center. If $\mathfrak{a}$ is not isomorphic to the three-dimensional Heisenberg algebra, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.* $\square$

The second lemma follows from a generalization of the proofs of [4, Lemma 4.2 and a part of Lemma 4.1] from $\mathbb{R}$ and $\mathbb{C}$ to arbitrary ground fields of characteristic zero (for a another generalization of the latter see also [6, Theorem 1]). In fact, the proofs in [4] remain valid in the more general setting.

Lemma 2. *Let $\mathfrak{a}$ be a finite-dimensional centerless solvable Lie algebra over a field $\mathbb{F}$ of characteristic zero. If $[\mathfrak{a}, \mathfrak{a}]$ is non-abelian or $\dim_{\mathbb{F}}[\mathfrak{a}, \mathfrak{a}] \geq 3$, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.* $\square$

Remark. The proof of [4, Lemma 4.2] is even valid for any field of characteristic $\neq 2$, but the proof of [4, Lemma 4.1] uses in an essential way that the ground field has characteristic zero.

Finally, in the non-solvable case we will need the following result.

Lemma 3. *Let $\mathfrak{a}$ be a finite-dimensional non-solvable Lie algebra over a field $\mathbb{F}$ of characteristic zero. If $\mathfrak{a}$ is not three-dimensional simple, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.*

Proof. Suppose that $\mathfrak{a}$ does not admit any non-trivial triangular Lie bialgebra structure. Since the ground field is assumed to have characteristic zero, the Levi decomposition theorem (see [9, p. 91]) yields the existence of a semisimple subalgebra $\mathfrak{l}$ of $\mathfrak{a}$ (a so-called *Levi factor* of $\mathfrak{a}$) such that $\mathfrak{a}$ is the semidirect product of $\mathfrak{l}$ and its solvable radical $\mathrm{Solv}(\mathfrak{a})$. Because $\mathfrak{a}$ is not solvable, the levi factor $\mathfrak{l}$ is non-zero.

Since $\mathbb{F}$ is infinite, $\mathfrak{l}$ contains a Cartan subalgebra $\mathfrak{h}$ (see [1, Corollary 1.2]). Let $\overline{\mathbb{F}}$ denote the algebraic closure of the ground field and set $\overline{\mathfrak{l}} := \mathfrak{l} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ as well as $\overline{\mathfrak{h}} := \mathfrak{h} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Then $\overline{\mathfrak{h}}$ is also a Cartan subalgebra of $\overline{\mathfrak{l}}$. According to the structure theory of finite-dimensional semisimple Lie algebras over algebraically closed fields of characteristic zero (cf. [9, Section IV.1]), $\overline{\mathfrak{h}}$ is abelian and $\overline{\mathfrak{l}}$ is the direct sum of $\overline{\mathfrak{h}}$ and the one-dimensional root spaces $\overline{\mathfrak{l}}_\alpha$ with $0 \neq \alpha \in \mathcal{R}$, where $\mathcal{R}$ denotes the set of roots of $\overline{\mathfrak{l}}$ relative to $\overline{\mathfrak{h}}$.

Suppose that $\dim_{\overline{\mathbb{F}}} \overline{\mathfrak{h}} \geq 2$. Then choose two linearly independent elements $h, h' \in \mathfrak{h}$ and define $r := h \wedge h'$. Clearly $\mathrm{CYB}(r) = 0$, and it is obvious that $\delta_r \neq 0$ if and only if $\overline{\delta_r} := \delta_r \otimes \mathrm{id}_{\overline{\mathbb{F}}} \neq 0$. Furthermore, set $\overline{h} := h \otimes 1_{\overline{\mathbb{F}}}$ resp. $\overline{h'} := h' \otimes 1_{\overline{\mathbb{F}}}$ and choose $\alpha \in \mathcal{R}$ with $\alpha(\overline{h}) \neq 0$. Then for every root vector $X \in \overline{\mathfrak{l}}_\alpha$ we have

$$\overline{\delta_r}(X) = \alpha(\overline{h})\overline{h'} \wedge X - \alpha(\overline{h'})\overline{h} \wedge X \neq 0.$$

Hence we can assume that $\dim_{\overline{\mathbb{F}}} \overline{\mathfrak{h}} = 1$, i.e., $\overline{\mathfrak{l}} \cong \mathfrak{sl}_2(\overline{\mathbb{F}})$. In particular, $\mathfrak{l}$ is three-dimensional simple.

Suppose now that $\mathfrak{s} := \mathrm{Solv}(\mathfrak{a}) \neq 0$. Without loss of generality we can assume that $\overline{\mathfrak{h}} = \overline{\mathbb{F}}\,\overline{h}$ with $\overline{h} := h \otimes 1_{\overline{\mathbb{F}}}$. Then it is well-known from the representation theory of $\mathfrak{sl}_2(\overline{\mathbb{F}})$ that either the weight space $(\mathfrak{s} \otimes_{\mathbb{F}} \overline{\mathbb{F}})_0 \cong \mathfrak{s}_0 \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ or the weight space $(\mathfrak{s} \otimes_{\mathbb{F}} \overline{\mathbb{F}})_1 \cong \mathfrak{s}_1 \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is non-zero (cf. [8, p. 33]). Let s be a non-zero weight vector in $\mathfrak{s}$ of weight 0 or 1 and define $r := h \wedge s$. Then by virtue of [11, Theorem 3.2], $\mathrm{CYB}(r) = 0$ and as above it is enough to show that $\overline{\delta_r} \neq 0$. Choose $E \in \overline{\mathfrak{l}}$ such that $[\overline{h}, E] = 2E$. (Note that this means in particular that $\overline{h}$ and E are *linearly independent* over $\overline{\mathbb{F}}$.) Then

$$\overline{\delta_r}(E) = -2E \wedge \overline{s} + \overline{h} \wedge [E, \overline{s}] \neq 0,$$

where $\overline{s} := s \otimes 1_{\overline{\mathbb{F}}}$. Hence $\mathrm{Solv}(\mathfrak{a}) = 0$ and thus $\mathfrak{a}$ is three-dimensional simple. $\square$

§3. Quaternionic Lie Bialgebras

In this section let $\mathbb{F}$ be an arbitrary field of characteristic $\neq 2$. In order to characterize those finite-dimensional non-solvable Lie algebras which admit non-trivial triangular Lie bialgebra structures, we need to consider a certain class of four-dimensional unital associative algebras.

Let α, β be non-zero elements of $\mathbb{F}$. Then the four-dimensional vector space

$$(\alpha, \beta)_{\mathbb{F}} := \mathbb{F}1 \oplus \mathbb{F}i \oplus \mathbb{F}j \oplus \mathbb{F}k$$

is an associative $\mathbb{F}$-algebra with unity element 1 and the defining relations

$$i^2 = -\alpha \cdot 1, \quad j^2 = -\beta \cdot 1, \quad ij = k = -ji.$$

One important example are *Hamilton's quaternions* which arise as $(1,1)_{\mathbb{R}}$. Therefore any algebra $(\alpha, \beta)_{\mathbb{F}}$ with $0 \neq \alpha, \beta \in \mathbb{F}$ is called a *quaternion algebra* over $\mathbb{F}$. Note that the assumption $\alpha \neq 0 \neq \beta$

assures that i and j (and thus also k) are not nilpotent. In fact, $\alpha \neq 0 \neq \beta$ implies that $(\alpha, \beta)_{\mathbb{F}}$ is a *central simple* algebra (cf. [12, Lemma 1.6] or [13, Lemma 11.15 in Chapter 2]).

If A is an associative algebra, then A is also a Lie algebra via the Lie bracket defined by $[x, y] := xy - yx$ for every $x, y \in A$ which will be denoted by $\mathcal{L}(A)$. For us the following well-known result will be useful (see [14, Corollary 1.6.2]).

Lemma 4. *If $\mathfrak{a}$ is a three-dimensional simple Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic $\neq 2$, then there exists a quaternion algebra Q such that $\mathfrak{a}$ is isomorphic to $\mathcal{L}(Q)/\mathbb{F}1_Q$.* $\square$

Set $[\alpha, \beta]_{\mathbb{F}} := \mathcal{L}((\alpha, \beta)_{\mathbb{F}})/\mathbb{F}1$ and let e_1 denote the residue class of $\frac{1}{2}i$, let e_2 denote the residue class of $\frac{1}{2}j$, and let e_3 denote the residue class of $\frac{1}{2}k$ in $[\alpha, \beta]_{\mathbb{F}}$. The Lie algebra $[\alpha, \beta]_{\mathbb{F}}$ is called a *quaternionic Lie algebra* over $\mathbb{F}$ and we have

$$[\alpha, \beta]_{\mathbb{F}} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus \mathbb{F}e_3$$

with the following Lie brackets

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = \beta e_1, \quad [e_3, e_1] = \alpha e_2. \tag{3}$$

Remark. According to (3), every quaternionic Lie algebra is perfect which in turn implies that every quaternionic Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2$ is simple (cf. [14, 3(d), p. 34]).

A three-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is called *split* if $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{F})$ and *non-split* otherwise. Similarly, a central simple algebra over $\mathbb{F}$ is called *split* if it is isomorphic to $\mathrm{Mat}_n(\mathbb{F})$ for some positive integer n and *non-split* otherwise.

Example 1. Note that $(-1, -1)_{\mathbb{F}}$ is isomorphic to $\mathrm{Mat}_2(\mathbb{F})$ (cf. the proof of [13, Corollary 11.14 in Chapter 2]). Let E_{ij} denote the 2×2 matrix having a 1 in the ij-th entry and 0's otherwise. Moreover, set $1 := E_{11} + E_{22}$. Then the residue classes H, E, and F of $E_{11} - E_{22}$, E_{12}, and E_{21}, respectively, in $[-1, -1]_{\mathbb{F}} = \mathfrak{gl}_2(\mathbb{F})/\mathbb{F}1$ are linearly independent over $\mathbb{F}$ and satisfy the relations $[H, E] = 2E$, $[H, F] = -2F$, as well as $[E, F] = H$. Consequently, $[-1, -1]_{\mathbb{F}} \cong \mathfrak{sl}_2(\mathbb{F})$.

Example 2. Consider the *non-split* three-dimensional simple real Lie algebra $\mathfrak{su}(2)$ (which can be realized as the cross product on three-dimensional euclidean space). Then $\mathfrak{su}(2) \cong [1, 1]_{\mathbb{F}}$ and it is well-known that $\mathfrak{su}(2)$ is the only *non-split* three-dimensional simple real Lie algebra (up to isomorphism). Note that this corresponds to Frobenius' classical result that Hamilton's quaternion algebra $(1, 1)_{\mathbb{R}}$ is the only non-split central simple $\mathbb{R}$-algebra (up to isomorphism) (cf. e.g. the argument after [12, Corollary 1.7]).

The following result determines exactly which quaternionic Lie algebras admit non-trivial (quasi-)triangular Lie bialgebra structures.

Proposition 1. *Let $[\alpha, \beta]_{\mathbb{F}}$ be a quaternionic Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2$. Then the following statements are equivalent:*

(a) $[\alpha, \beta]_{\mathbb{F}}$ *admits a non-trivial triangular Lie bialgebra structure.*

(b) $[\alpha, \beta]_{\mathbb{F}}$ *admits a non-trivial quasi-triangular Lie bialgebra structure.*

(c) $[\alpha, \beta]_{\mathbb{F}}$ *is isomorphic to the split three-dimensional simple Lie algebra* $\mathfrak{sl}_2(\mathbb{F})$.

Proof. Since the implication (a)$\Longrightarrow$(b) is trivial and the implication (c)$\Longrightarrow$(a) is an immediate consequence of [11, Theorem 3.2], it is enough to show the implication (b)$\Longrightarrow$(c).

In order to prove the remaining implication, let us consider the quaternionic Lie algebra $\mathfrak{g} := [\alpha, \beta]_{\mathbb{F}}$ with $0 \neq \alpha, \beta \in \mathbb{F}$. Then it follows from a straightforward computation that

$$(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} = \mathbb{F}(\beta e_1 \otimes e_1 + \alpha e_2 \otimes e_2 + e_3 \otimes e_3).$$

Hence, $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies $r + \tau(r) \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ if and only if

$$r = \eta(\beta e_1 \otimes e_1 + \alpha e_2 \otimes e_2 + e_3 \otimes e_3) + \eta_{12} e_1 \wedge e_2 + \eta_{23} e_2 \wedge e_3 + \eta_{31} e_3 \wedge e_1,$$

for some $\eta, \eta_{12}, \eta_{23}, \eta_{31} \in \mathbb{F}$. By virtue of [3, Remark 2 after the proof of Lemma 2.1.3], we have

$$\mathrm{CYB}(r) = \eta^2 \mathrm{CYB}(\beta e_1 \otimes e_1 + \alpha e_2 \otimes e_2 + e_3 \otimes e_3) + \mathrm{CYB}(\eta_{12} e_1 \wedge e_2 + \eta_{23} e_2 \wedge e_3 + \eta_{31} e_3 \wedge e_1).$$

A straightforward computation yields

$$\mathrm{CYB}(\beta e_1 \otimes e_1 + \alpha e_2 \otimes e_2 + e_3 \otimes e_3) = (\alpha\beta) e_1 \wedge e_2 \wedge e_3.$$

Another straightforward computation shows that $\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}$ is an *orthogonal* basis of $\mathfrak{g} \wedge \mathfrak{g}$ with respect to Drinfel'd's Poisson superbracket $\{\cdot, \cdot\}$ (cf. [6, Proposition 2]) if we identify $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ with $\mathbb{F}$ via $e_1 \wedge e_2 \wedge e_3 \mapsto 1_{\mathbb{F}}$. According to [6, Proposition 2 and (3)], we obtain

$$\mathrm{CYB}(\eta_{12} e_1 \wedge e_2 + \eta_{23} e_2 \wedge e_3 + \eta_{31} e_3 \wedge e_1) = (\eta_{12}^2 + \beta\eta_{23}^2 + \alpha\eta_{31}^2) e_1 \wedge e_2 \wedge e_3. \tag{4}$$

Combining the last two results, we conclude that

$$\mathrm{CYB}(r) = (\alpha\beta\eta^2 + \eta_{12}^2 + \beta\eta_{23}^2 + \alpha\eta_{31}^2) e_1 \wedge e_2 \wedge e_3.$$

Hence $\mathrm{CYB}(r) = 0$ if and only if $(\eta_{12}, \eta_{31}, \eta_{23}, \eta) \in \mathbb{F}^4$ is an isotropic vector of the quadratic form $W^2 + \alpha X^2 + \beta Y^2 + \alpha\beta Z^2$. But the latter is just the norm form of the quaternion algebra $(\alpha, \beta)_{\mathbb{F}}$ and thus [13, Corollary 11.10 in Chapter 2] implies that the CYBE for $\mathfrak{g}$ has a non-zero solution with $\mathfrak{g}$-invariant symmetric part if and only if $(\alpha, \beta)_{\mathbb{F}} \cong (-1, -1)_{\mathbb{F}}$. Finally, Example 1 shows that in the latter case $[\alpha, \beta]_{\mathbb{F}} \cong [-1, -1]_{\mathbb{F}} \cong \mathfrak{sl}_2(\mathbb{F})$. $\square$

Remark. It follows from Lemma 4 and Proposition 1 that the classical Yang-Baxter equation for a three-dimensional simple Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic $\neq 2$ has a non-zero solution with $\mathfrak{g}$-invariant symmetric part if and only if $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{F})$. In the latter case, all solutions of the classical Yang-Baxter equation with invariant symmetric part are well-known (cf. [2] and [3, Example 2.1.8]).

Let us conclude this section by relating the classical Yang-Baxter operator in the case of a three-dimensional simple Lie algebra to the determinant. Let $\mathfrak{g}$ be a three-dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2$. Then the Lie bracket of $\mathfrak{g}$ induces a mapping γ from $\mathfrak{g} \wedge \mathfrak{g}$ into $\mathfrak{g}$ which is bijective since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $\dim_{\mathbb{F}} \mathfrak{g} = 3$. Since in this case $\dim_{\mathbb{F}} \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} = 1$, there is also a canonical bijection ι from $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ onto $\mathbb{F}$. In particular, $\iota \circ \mathrm{CYB}$ is a *quadratic form* with associated symmetric bilinear form $\iota \circ \{\cdot, \cdot\}$, where $\{\cdot, \cdot\}$ denotes Drinfel'd's Poisson superbracket (cf. [6, Proposition 2]).

According to Lemma 4, $\mathfrak{g} \cong [\alpha, \beta]_{\mathbb{F}}$ for some $0 \neq \alpha, \beta \in \mathbb{F}$. Let $\sqrt{\alpha}$ and $\sqrt{\beta}$ denote solutions of $X^2 = \alpha$ and $X^2 = \beta$, respectively in the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. Moreover, let ζ denote a solution of $X^2 + 1 = 0$ in $\overline{\mathbb{F}}$. Consider the quaternion algebra $(\alpha, \beta)_{\mathbb{F}}$. Then the mapping defined by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} \sqrt{\alpha}\zeta & 0 \\ 0 & -\sqrt{\alpha}\zeta \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & \sqrt{\beta} \\ -\sqrt{\beta} & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & \sqrt{\alpha\beta}\zeta \\ \sqrt{\alpha\beta}\zeta & 0 \end{pmatrix}$$

defines a *two-dimensional faithful* representation ρ of $(\alpha,\beta)_{\mathbb{F}}$ over $\overline{\mathbb{F}}$. In fact, ρ is an isomorphism $(\alpha,\beta)_{\mathbb{F}} \cong \mathrm{Mat}_2(\overline{\mathbb{F}})$ of associative $\mathbb{F}$-algebras. Consequently, the mapping defined by

$$e_1 \mapsto \frac{1}{2}\begin{pmatrix} \sqrt{\alpha}\zeta & 0 \\ 0 & -\sqrt{\alpha}\zeta \end{pmatrix}, \quad e_2 \mapsto \frac{1}{2}\begin{pmatrix} 0 & \sqrt{\beta} \\ -\sqrt{\beta} & 0 \end{pmatrix}, \quad e_3 \mapsto \frac{1}{2}\begin{pmatrix} 0 & \sqrt{\alpha\beta}\zeta \\ \sqrt{\alpha\beta}\zeta & 0 \end{pmatrix}$$

defines a *two-dimensional faithful* representation ρ of $\mathfrak{g} \cong [\alpha,\beta]_{\mathbb{F}}$ over $\overline{\mathbb{F}}$. An easy calculation shows that

$$(\det\circ\rho\circ\gamma)(\eta_{12}e_1\wedge e_2 + \eta_{23}e_2\wedge e_3 + \eta_{31}e_3\wedge e_1) = \frac{1}{4}\alpha\beta(\eta_{12}^2 + \beta\eta_{23}^2 + \alpha\eta_{31}^2).$$

Comparing this with (4) yields the following result which was observed for $\mathfrak{sl}_2$ in [3, Example 2.1.8] and for $\mathfrak{su}(2)$ in [6, Remark after Example 1].

Proposition 2. *Let $\mathfrak{g} \cong [\alpha,\beta]_{\mathbb{F}}$ be a three-dimensional simple Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic $\neq 2$ with $0 \neq \alpha,\beta \in \mathbb{F}$. Then the diagram*

$$\begin{array}{ccc} \mathfrak{g}\wedge\mathfrak{g} & \xrightarrow{\mathrm{CYB}} & \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g} \\ \gamma\downarrow & & \downarrow\iota \\ \mathfrak{g} & \xrightarrow{\frac{4}{\alpha\beta}\det\circ\rho} & \mathbb{F} \end{array}$$

is commutative. □

§4. Main Results

Let us consider the three-dimensional *solvable* Lie algebra

$$\mathfrak{s}_\Lambda(\mathbb{F}) = \mathbb{F}h \oplus \mathbb{F}s_1 \oplus \mathbb{F}s_2;$$

$$[h,s_1] = \lambda_{11}s_1 + \lambda_{12}s_2, \quad [h,s_2] = \lambda_{21}s_1 + \lambda_{22}s_2, \quad [s_1,s_2] = 0,$$

where $\Lambda := (\lambda_{ij})_{1\le i,j\le 2}$ is an element of $\mathrm{Mat}_2(\mathbb{F})$.

If $\Lambda = 0$, then $\mathfrak{s}_\Lambda(\mathbb{F})$ is abelian. If $\Lambda \neq 0$ is *singular*, then $\mathfrak{s}_\Lambda(\mathbb{F})$ is either isomorphic to the three-dimensional Heisenberg algebra or isomorphic to the (trivial) one-dimensional central extension of the two-dimensional non-abelian Lie algebra. Finally, if $\Lambda \in \mathrm{GL}_2(\mathbb{F})$, then $C(\mathfrak{a}) = 0$ and $\dim_{\mathbb{F}}[\mathfrak{a},\mathfrak{a}] = 2$. (In fact, $\Lambda \in \mathrm{GL}_2(\mathbb{F})$ if and only if $C(\mathfrak{a}) = 0$ if and only if $\dim_{\mathbb{F}}[\mathfrak{a},\mathfrak{a}] = 2$.)

Remark. It is elementary to show that every *non-simple* three-dimensional Lie algebra is isomorphic to $\mathfrak{s}_\Lambda(\mathbb{F})$ for a suitable choice of $\Lambda \in \mathrm{Mat}_2(\mathbb{F})$ (cf. e.g. [9, Section I.4] or [14, Section 1.6]).

Let $\mathbb{F}^2 := \{\xi^2 \mid \xi \in \mathbb{F}\}$ denote the squares in $\mathbb{F}$. We can now state the main result of this paper.

Theorem. *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then the following statements are equivalent:*

(a) $\mathfrak{a}$ *admits a non-trivial triangular Lie bialgebra structure.*

(b) $\mathfrak{a}$ *admits a non-trivial quasi-triangular Lie bialgebra structure.*

(c) $\mathfrak{a}$ *is non-abelian and neither a non-split three-dimensional simple Lie algebra over $\mathbb{F}$ nor isomorphic to the three-dimensional Heisenberg algebra $\mathfrak{h}_1(\mathbb{F})$ or $\mathfrak{s}_\Lambda(\mathbb{F})$ with $\mathrm{tr}(\Lambda) = 0$ and $-\det(\Lambda) \notin \mathbb{F}^2$.*

Proof. Since the implication (a)$\Longrightarrow$(b) is trivial, it is enough to show the implications (b)$\Longrightarrow$(c) and (c)$\Longrightarrow$(a).

(b)$\Longrightarrow$(c): According to Lemma 4 and Proposition 1, a non-split three-dimensional simple Lie algebra does not admit any non-trivial quasi-triangular Lie bialgebra structure. Since it is clear that every coboundary Lie bialgebra structure on an abelian Lie algebra is trivial, it suffices to prove that $\mathfrak{h}_1(\mathbb{F})$ as well as $\mathfrak{s}_\Lambda(\mathbb{F})$ with $\mathrm{tr}(\Lambda) = 0$ and $-\det(\Lambda) \notin \mathbb{F}^2$ do not admit any non-trivial quasi-triangular Lie bialgebra structure. For $\mathfrak{h}_1(\mathbb{F})$ this was already done in [6, Example 2]. Let us now consider $\mathfrak{s} := \mathfrak{s}_\Lambda(\mathbb{F})$ where $\det(\Lambda) \neq 0$. Then a straightforward computation yields

$$(\mathfrak{s} \otimes \mathfrak{s})^{\mathfrak{s}} = \begin{cases} \mathbb{F}(s_1 \wedge s_2) \oplus \mathbb{F}[\lambda_{21}(s_1 \otimes s_1) - \lambda_{11}(s_1 \otimes s_2 + s_2 \otimes s_1) - \lambda_{12}(s_2 \otimes s_2)] & \text{if } \mathrm{tr}(\Lambda) = 0 \\ 0 & \text{if } \mathrm{tr}(\Lambda) \neq 0 \end{cases}$$

Consider now a 2-tensor $r = r_0 + r_*$ with $\mathfrak{s}$-invariant symmetric part r_0 and skew-symmetric part r_*. Because of $[s_1, s_2] = 0$, we conclude from [3, Remark 2 after the proof of Lemma 2.1.3] that

$$\mathrm{CYB}(r) = \mathrm{CYB}(r_0) + \mathrm{CYB}(r_*) = \mathrm{CYB}(r_*)$$

and

$$\delta_r(x) = x \cdot r = x \cdot r_0 + x \cdot r_* = x \cdot r_* = \delta_{r_*}(x)$$

for every $x \in \mathfrak{s}$. Consequently, $\mathfrak{s}$ admits a non-trivial quasi-triangular Lie bialgebra structure if and only if it admits a non-trivial triangular Lie bialgebra structure. Hence it will follow directly from the argument below that $\mathfrak{s}$ does *not* admit *any* non-trivial quasi-triangular Lie bialgebra structure unless $\mathrm{tr}(\Lambda) \neq 0$ or $\mathrm{tr}(\Lambda) = 0$ and $-\det(\Lambda) \in \mathbb{F}^2$.

(c)$\Longrightarrow$(a): Suppose that $\mathfrak{a}$ does not admit any non-trivial triangular Lie bialgebra structure. If $\mathfrak{a}$ is not solvable, then it follows from Lemma 3, Lemma 4, and Proposition 1 that $\mathfrak{a}$ is isomorphic to a non-split three-dimensional simple Lie algebra over $\mathbb{F}$.

If $\mathfrak{a}$ is solvable, then by virtue of Lemmas 1 and 2, we can assume for the rest of the proof that the center $C(\mathfrak{a})$ of $\mathfrak{a}$ is zero and $[\mathfrak{a}, \mathfrak{a}]$ is abelian of dimension at most 2.

Suppose that $\dim_{\mathbb{F}}[\mathfrak{a}, \mathfrak{a}] = 1$. Since $C(\mathfrak{a}) = 0$, for any non-zero element $e \in [\mathfrak{a}, \mathfrak{a}]$ there exists an element $a \in \mathfrak{a}$ such that $[a, e] \neq 0$. (Note that this means in particular that a and e are *linearly independent* over $\mathbb{F}$.) But because of $\dim_{\mathbb{F}}[\mathfrak{a}, \mathfrak{a}] = 1$, we have $[a, e] = \lambda e$ for some $0 \neq \lambda \in \mathbb{F}$. Set $r := a \wedge e \in \mathrm{Im}(\mathrm{id}_{\mathfrak{a}} - \tau)$. Then it follows from [11, Theorem 3.2] that r is a solution of the CYBE and

$$\delta_r(a) = [a, a] \wedge e + a \wedge [a, e] = \lambda \cdot (a \wedge e) = \lambda \cdot r \neq 0$$

implies that δ_r defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{a}$.

Hence we can assume from now on that $C(\mathfrak{a}) = 0$ and that $[\mathfrak{a}, \mathfrak{a}]$ is two-dimensional abelian. Since $\dim_{\mathbb{F}}[\mathfrak{a}, \mathfrak{a}] = 2$, there exist $s_1, s_2 \in \mathfrak{a}$ such that

$$[\mathfrak{a}, \mathfrak{a}] = \mathbb{F}s_1 \oplus \mathbb{F}s_2.$$

Next, we show that $\dim_{\mathbb{F}} \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] = 1$. Suppose to the contrary that $\dim_{\mathbb{F}} \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] \geq 2$. Because of $C(\mathfrak{a}) = 0$, there is an element $a \in \mathfrak{a}$ such that $[a, s_1] \neq 0$. In particular, $a \notin \mathbb{F}s_1 \oplus \mathbb{F}s_2$, i.e., a, s_1, and s_2 are linearly independent over $\mathbb{F}$. It follows from $\dim_{\mathbb{F}} \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] \geq 2$ that there also is an element $a' \in \mathfrak{a}$ such that a, a', s_1, and s_2 are linearly independent over $\mathbb{F}$. Moreover, for every $1 \leq i, j \leq 2$, there exist elements $\alpha_{ij}, \alpha'_{ij} \in \mathbb{F}$ such that

$$\begin{aligned} [a, s_1] &= \alpha_{11}s_1 + \alpha_{12}s_2, \quad & [a, s_2] &= \alpha_{21}s_1 + \alpha_{22}s_2, \\ [a', s_1] &= \alpha'_{11}s_1 + \alpha'_{12}s_2, \quad & [a', s_2] &= \alpha'_{21}s_1 + \alpha'_{22}s_2. \end{aligned}$$

If $\alpha_{12} = 0$, then $[a, s_1] = \alpha_{11} s_1 \neq 0$, and one can argue as above (for the case $\dim_{\mathbb{F}}[\mathfrak{a}, \mathfrak{a}] = 1$) that $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure. On the other hand, if $\alpha_{12} \neq 0$, let us set $h := \alpha'_{12} a - \alpha_{12} a'$ and $\lambda := \alpha'_{12}\alpha_{11} - \alpha_{12}\alpha'_{11}$. Then $h \notin [\mathfrak{a}, \mathfrak{a}]$ and $[h, s_1] = \lambda s_1$. If we now put $r := h \wedge s_1 \in \mathrm{Im}(\mathrm{id}_{\mathfrak{a}} - \tau)$, we see as before that r is a solution of the CYBE. Since $h \notin [\mathfrak{a}, \mathfrak{a}]$ and $[a, s_1] \neq 0$, we conclude that

$$\delta_r(a) = [a, h] \wedge s_1 + h \wedge [a, s_1] \neq 0.$$

Hence δ_r defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{a}$.

Finally, we can assume that $\mathfrak{a}$ is three-dimensional and $[\mathfrak{a}, \mathfrak{a}]$ is two-dimensional abelian. It follows that $\mathfrak{a} \cong \mathfrak{s}_\Lambda(\mathbb{F})$ with $\det(\Lambda) \neq 0$. Then we obtain for an arbitrary skew-symmetric 2-tensor

$$r = \omega s_1 \wedge s_2 + \xi_1 h \wedge s_1 + \xi_2 h \wedge s_2$$

with $\omega, \xi_1, \xi_2 \in \mathbb{R}$ that

$$\mathrm{CYB}(r) = [\lambda_{12}\xi_1^2 - (\lambda_{11} - \lambda_{22})\xi_1\xi_2 - \lambda_{21}\xi_2^2] \cdot h \wedge s_1 \wedge s_2.$$

If $\mathrm{tr}(\Lambda) \neq 0$, then – as already established in the proof of the implication (b)$\Longrightarrow$(c) – there is *no* non-zero $\mathfrak{s}_\Lambda(\mathbb{F})$-invariant 2-tensor. Consequently, $r := s_1 \wedge s_2$ defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{s}_\Lambda(\mathbb{F})$.

On the other hand, if $\mathrm{tr}(\Lambda) = 0$, then the discriminant of the relevant homogeneous quadratic equation

$$\lambda_{12} X_1^2 - (\lambda_{11} - \lambda_{22}) X_1 X_2 - \lambda_{21} X_2^2 = 0$$

is the *negative* of $\det(\Lambda)$. Hence $\mathfrak{s}_\Lambda(\mathbb{F})$ admits a non-trivial triangular Lie bialgebra structure if and only if $-\det(\Lambda) \in \mathbb{F}^2$. $\square$

As an immediate consequence of the theorem we obtain the following existence result:

Corollary 1. *If $\mathfrak{a}$ is a finite-dimensional non-abelian Lie algebra over a field $\mathbb{F}$ of characteristic zero with $\dim_{\mathbb{F}} \mathfrak{a} \neq 3$, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.* $\square$

We conclude with the following generalization of the main result of [4] from $\mathbb{R}$ and $\mathbb{C}$ to arbitrary ground fields of characteristic zero (for another generalization see also the remark after [6, Theorem 4]).

Corollary 2. *Every finite-dimensional non-abelian Lie algebra over an arbitrary field $\mathbb{F}$ of characteristic zero admits a non-trivial coboundary Lie bialgebra structure.*

Proof. According to the theorem, we only have to prove the existence of a non-trivial coboundary Lie bialgebra structure for a non-split three-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the three-dimensional Heisenberg algebra $\mathfrak{h}_1(\mathbb{F})$, and the three-dimensional solvable Lie algebra $\mathfrak{s}_\Lambda(\mathbb{F})$ with $\mathrm{tr}(\Lambda) = 0$ and $-\det(\Lambda) \notin \mathbb{F}^2$.

First, let us consider a non-split three-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$. By virtue of Lemma 4, $\mathfrak{g}$ is a quaternionic Lie algebra $[\alpha, \beta]_{\mathbb{F}}$ for some $0 \neq \alpha, \beta \in \mathbb{F}$. Set $r := e_1 \wedge e_2$. Then it follows from [6, Proposition 2] that $\mathrm{CYB}(r) = 2e_1 \wedge e_2 \wedge e_3$ and thus a straightforward computation shows that $\mathrm{CYB}(r)$ is $\mathfrak{g}$-invariant. Because of $\delta_r(e_1) = 2e_1 \wedge e_3 \neq 0$, the skew-symmetric 2-tensor r defines a non-trivial coboundary Lie bialgebra structure on $\mathfrak{g}$.

In the case of the three-dimensional Heisenberg algebra $\mathfrak{h} := \mathfrak{h}_1(\mathbb{F})$ set $r := p \wedge q$. It was shown in the proof of [6, Theorem 3] that $\mathrm{CYB}(r) = p \wedge q \wedge \hbar$ which clearly is $\mathfrak{h}$-invariant. Since

$$\delta_r(p) = p \wedge \hbar \neq 0 \neq q \wedge \hbar = \delta_r(q)\,,$$

the skew-symmetric 2-tensor r defines a non-trivial coboundary Lie bialgebra structure on $\mathfrak{h}$.

Finally, consider $\mathfrak{s} := \mathfrak{s}_\Lambda(\mathbb{F})$ with $\mathrm{tr}(\Lambda) = 0$ and $-\det(\Lambda) \notin \mathbb{F}^2$. Set $r_1 := h \wedge s_1$ and $r_2 := h \wedge s_2$. Then it follows from [6, Proposition 2] that $\mathrm{CYB}(r_1) = \lambda_{12} h \wedge s_1 \wedge s_2$ and $\mathrm{CYB}(r_2) = -\lambda_{21} h \wedge s_1 \wedge s_2$. But obviously, $s_1 \cdot (h \wedge s_1 \wedge s_2) = 0$, $s_2 \cdot (h \wedge s_1 \wedge s_2) = 0$, and $h \cdot (h \wedge s_1 \wedge s_2) = \mathrm{tr}(\Lambda) h \wedge s_1 \wedge s_2$. Hence $\mathrm{tr}(\Lambda) = 0$ implies that $\mathrm{CYB}(r_1)$ and $\mathrm{CYB}(r_2)$ are both $\mathfrak{s}$-invariant. On the other hand,

$$\delta_{r_1}(h) = \lambda_{11} h \wedge s_1 + \lambda_{12} h \wedge s_2$$

and

$$\delta_{r_2}(h) = \lambda_{21} h \wedge s_1 + \lambda_{22} h \wedge s_2$$

show that at least one of the skew-symmetric 2-tensors r_1 and r_2 defines a non-trivial coboundary Lie bialgebra structure on $\mathfrak{s}$ if $\mathrm{tr}(\Lambda) = 0$ and $\Lambda \neq 0$. $\square$

References

[1] D. W. Barnes: On Cartan subalgebras of Lie algebras, *Math. Z.* **101** (1967), 350–355.

[2] A. A. Belavin and V. G. Drinfel'd: Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funct. Anal. Appl.* **16** (1982), 159–180.

[3] V. Chari and A. Pressley: *A Guide to Quantum Groups*, 1st paperback ed., with corrections, Cambridge University Press, Cambridge, 1995.

[4] V. de Smedt: Existence of a Lie bialgebra structure on every Lie algebra, *Lett. Math. Phys.* **31** (1994), 225–231.

[5] V. G. Drinfel'd: Quantum groups, in: *Proc. Internat. Congr. Math., Berkeley, 1986* (ed. A. M. Gleason), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.

[6] J. Feldvoss: Existence of triangular Lie bialgebra structures, *J. Pure Appl. Algebra* **134** (1999), 1–14.

[7] J. Feldvoss: Existence of solutions of the classical Yang-Baxter equation for a real Lie algebra, *Abh. Math. Sem. Univ. Hamburg* **71** (2001), 297–304.

[8] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*, Third Printing, Revised, Graduate Texts in Mathematics, Vol. **9**, Springer-Verlag, New York/Heidelberg/Berlin, 1980.

[9] N. Jacobson: *Lie Algebras*, Dover Publications Inc., New York, 1979 (unabridged, corrected republication of the original edition from 1962).

[10] W. Michaelis: Lie coalgebras, *Adv. Math.* **38** (1980), 1–54.

[11] W. Michaelis: A class of infinite-dimensional Lie bialgebras containing the Virasoro algebra, *Adv. Math.* **107** (1994), 365–392.

[12] R. S. Pierce: *Associative Algebras*, Graduate Texts in Mathematics, Vol. **88**, Springer-Verlag, New York/Heidelberg/Berlin, 1982.

[13] W. Scharlau: *Quadratic and Hermitian Forms*, Grundlehren der mathematischen Wissenschaften, Vol. **270**, Springer-Verlag, Berlin/Heidelberg/New York/Tokio, 1985.

[14] H. Strade and R. Farnsteiner: *Modular Lie Algebras and Their Representations*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. **116**, Marcel Dekker, New York and Basel, 1988.